3

Matrices

Structure

3.1. Introduction. In 1857, Arthur Cayley, formulated the general theory of matrices. In the study of mathematics matrices have the most important place and found useful in many branches of science, engineering, economics and commerce. This chapter contains many important results related to matrices, their addition subtraction, multiplication, various types and their realtions.

3.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Matrices.

(ii) Various types of matrices.

(iii) Algebra of matrices.

3.1.2. Keywords. Matrix, Comparable Matrices, Symmetric Matrix, Non-Symmetric Matrix.

3.2. Matrices.

3.2.1. Matrix. A rectangular representation of numbers (data) or functions is known as a matrix. A matrix is always represented by capital letters *A*, *B*, *C, …* etc.

For example $A=\begin{bmatrix}1 & 1 & 6\\ 2 & 0 & 3\end{bmatrix}$ is a matrix.

In a matrix, horizontal lines are called **rows** and vertical lines are called **columns**. For example**,**

$$A=\underbrace{\begin{bmatrix}1 & 1 & 6\\ 2 & 0 & 3\end{bmatrix}}_{\substack{\downarrow\ \downarrow\ \downarrow\\ \text{Columns}}}\ \left.\begin{matrix}\rightarrow\\ \rightarrow\end{matrix}\right\}\text{Rows}$$

A matrix may have any number of rows and columns.

3.2.2. Order of matrix. By stating that A is a matrix of order m x n, we mean that the matrix A is having m rows and n columns.

Generally, a matrix is represented as $A=\left[a_{ij}\right]_{mxn}$, which is a matrix of order m x n, having m rows and n columns. a_{ij} ’s are known as elements of the matrix A. In particular a_{ij} is the jth entry in the ith row.

3.2.3. Types of Matrices.

Here we are discussing some useful types of matrices:

3.2.4. Square matrix. A matrix of order $m \times n$ is called a square matrix if $m = n$, that is, if the number of rows is equal to number of columns

For example**.** $A=\begin{bmatrix}1 & 1 & 8\\ 0 & 0 & 0\\ 2 & 1 & 5\end{bmatrix}$ is a square matrix of order 3.

The elements $a_{11}, a_{22}, \ldots$ are called the diagonal elements of matrix A . Thus, those a_{ij} for which $i = j$ are called diagonal elements. Rest of the elements are called the non-diagonal elements of square matrix A, that is, those a_{ij} for which $i \neq j$.

3.2.5. Row matrix. A matrix having single row and any number of columns is called a row matrix. For example, $A=[1\ \ 0\ \ 4\ \ 7\ \ 5\ \ 6]$ is a row matrix of order 1×6.

3.2.6. Column matrix. A matrix having single column and any number of rows is called column matrix.

For example, $A=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$ is a column matrix of order 3×1.

3.2.7. Zero or null matrix. A matrix is said to be a zero or null matrix if all its elements are zero. Usually it is denoted by O. For example, $A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is a zero matrix.

3.2.8. Diagonal matrix. A square matrix in which all non-diagonal elements are zero is called a diagonal matrix. So, $A=\left[a_{ij}\right]_{m\times n}$ is a diagonal matrix if a_{ij}=0 for $i \neq j$.

For example, $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix of order 3 × 3

It should be noted that the diagonal elements in a diagonal matrix may or may not be zero. Further, a diagonal matrix of order n can be denoted as $A = diag.[a_{11} \quad a_{22} \quad . \quad . \quad . \quad a_{nn}]$.

3.2.9. Scalar matrix. A diagonal matrix in which all its diagonal elements are equal is called a scalar matrix**.** For example**,** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix of order 3 × 3. Thus, a diagonal matrix of order n can be denoted as $A = diag.[a \quad a \quad . \quad . \quad . \quad a]$.

Note. All square zero matrices are always diagonal as well as scalar matrix**.**

3.2.10. Unit matrix or Identity matrix. A scalar matrix with all entries '1' is called an identity matrix. Usually a unit matrix is denotes by I_n where n represents order of matrix.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3.2.11. Triangular Matrices.

(i) Upper triangular matrix. A matrix in which all elements below the principal diagonal are zero is called an upper triangular matrix. For example**,** $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 4 \end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

(ii) Lower triangular matrix. A matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix. For example**,** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 4 & 2 & 7 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3.

Remark. Diagonal matrices are upper triangular as well as lower triangular matrices.

3.2.12. Comparable matrices. Two matrices A and B are comparable if their orders are same, that is, if A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then A and B are comparable if $m = p$ and $n = q$

3.2.13. Equal matrices. Two matrices are equal if they are of same order and having same elements in the corresponding positions. For example, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 6 \end{bmatrix}$ then $a = 1$, $b = 4$, $c = 0$, $d = 6$.

3.2.14. Example. Construct a 2 × 3 matrix $A = [a_{ij}]$ whose element a_{ij} are given by

$$a_{ij} = \frac{(i + 2j)^2}{3}$$

Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

Given that $a_{ij} = \frac{(i+2j)^2}{3}$

Then, $a_{11} = \frac{[(1)+2(1)]^2}{3} = 3$, $a_{12} = \frac{[(1)+2(2)]^2}{3} = \frac{25}{3}$, $a_{13} = \frac{[(1)+2(3)]^2}{3} = \frac{49}{3}$

$a_{21} = \frac{[(2)+2(1)]^2}{3} = \frac{16}{3}$, $a_{22} = \frac{[(2)+2(2)]^2}{3} = 12$, $a_{23} = \frac{[(2)+2(3)]^2}{3} = \frac{64}{3}$

Therefore the required matrix is $A = \begin{bmatrix} 3 & \frac{25}{3} & \frac{49}{3} \\ \frac{16}{3} & 12 & \frac{64}{3} \end{bmatrix}$.

3.2.15. Exercise. If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ then find *x*, *y*, *z* and *w*.

Solution. Given that $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Comparing the corresponding elements, we obtain

$$x - y = -1$$

$$2x + z = 5$$

$$2x - y = 0$$

$$3z + w = 13$$

Subtracting first and third equation we can obtain

$$x = 1$$

Using $x = 1$ in first equation, we get

$$y = 2$$

Then from second equation, we get

$$2 + z = 5 \quad \Rightarrow \quad z = 3$$

Then from fourth equation, we get

$$9 + w = 13 \quad \Rightarrow \quad w = 4$$

Hence $x = 1$, $y = 2$, $z = 3$, $w = 4$

3.2.16. Exercise.

1. Construct a 3×3 matrix $A = [a_{ij}]$ whose element a_{ij} is given by $\frac{|-3+i+j|}{2}$.
2. Give an example of a matrix which is diagonal but not scalar.
3. For what values of *a* and *b* the following matrices are equal

$$A = \begin{bmatrix} 2a+1 & 3b \\ 5 & b^2 - 5b \end{bmatrix} \quad B = \begin{bmatrix} a+3 & b^2+2 \\ 5 & -6 \end{bmatrix}$$

4. Find the values of a and b if

   (i) $\begin{bmatrix} a & 5 \\ 6 & b \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} a+b & 8 \\ 6 & ab \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 6 & 8 \end{bmatrix}$

5. For what values of a and b are the matrices $A = \begin{bmatrix} a+3 & b^2 \\ 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2a+1 & 2b \\ 0 & b^2-5b \end{bmatrix}$ are equal?

Answer.

2. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. $a = 2$, $b = 2$

4. (i) $a = 7$, $b = 4$ (ii) $a = 2$, $b = 4$, or $a = 4$, $b = 2$,

5. $a = 2$, $b = 2$

3.3. Sum, Difference and Scalar Multiplication of Matrices.

3.3.1. Addition of Matrices.

If A and B are two matrices of same order, then their sum $A+B$ is obtained by adding the corresponding elements of A and B. Clearly order of $A + B$ is similar to that of A and B.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 3 \end{bmatrix}_{2\times 3}$. Then, $A+B = \begin{bmatrix} 4 & 2 & 4 \\ 4 & 7 & 6 \end{bmatrix}_{2\times 3}$.

Note. Addition of two or more matrices is defined only when they are comparable otherwise sum of two matrices does not exist.

3.3.2. Properties of matrix addition

If A and B are two matrices of same order then following properties holds:

(i) Matrix addition is commutative **that is,** , $A + B = B + A$

(ii) Matrix addition is associative **that is,** , $(A+B)+C = A+(B+C)$

3.3.3. Difference of matrices.

For two matrices A and B of the same order, then their difference A - B is obtained by subtracting the elements of B from the corresponding elements of A. Clearly order of A - B is similar to that of A and B.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 3 \end{bmatrix}_{2\times 3}$. Then, $A-B = \begin{bmatrix} -2 & 0 & -4 \\ 0 & -3 & 0 \end{bmatrix}_{2\times 3}$.

3.3.4. Scalar Multiplication and its Properties.

If A is any matrix of order $m \times n$ and k is any scalar then kA is obtained by multiplying every element of A with k and known as scalar multiple of A by k.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $k = 3$. Then, $kA = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 9 \end{bmatrix}_{2\times 3}$.

3.3.5. Properties of Scalar Multiplication.

If A and B are two matrices of same order and k is any scalar then following properties holds**.**

(i) $k(A+B) = kA + kB$ **(ii)** $kl(A) = k(lA) = l(kA)$

(iii) $(k+l)A = kA + lA$

3.3.6. Example. If A = $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 4 \end{bmatrix}$ **and B =** $\begin{bmatrix} 0 & 1 & 4 \\ -2 & 3 & -2 \end{bmatrix}$ **, then find *A* + *B* and *A* − *B*.**

Solution. Since the order of A and B are same, therefore addition and subtraction are possible. Then by definition

$$A + B = \begin{bmatrix} 1+0 & 5+1 & 6+4 \\ 2-2 & 4+3 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 10 \\ 0 & 7 & 2 \end{bmatrix}$$

and $$A - B = \begin{bmatrix} 1-0 & 5-1 & 6-4 \\ 2+2 & 4-3 & 4+2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 6 \end{bmatrix}.$$

3.3.7. Example. If *A* = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ **and *B* =** $\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$**, then find a matrix C such that** $A+B+C$ **is a zero matrix.**

Solution. Given that A + B + C = 0

$$\Rightarrow \quad C = -A - B = -(A + B).$$

Here, $$A + B = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

Therefore, $$C = (-1)(B + A) = \begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}.$$

3.3.8. Example. Find *x* and *y*, if $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$**.**

Solution. Given that $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

which implies $$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

which implies $$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Then, we have

$2x + 3 = 7$ and $2y - 4 = 14$

which implies $2x = 4$ and $2y = 18$

or $x = 2$ and $y = 9$

So $x = 2, y = 5$.

3.3.9. Exercise.

1. If $A=\begin{bmatrix}2 & 3\\ 4 & 6\end{bmatrix}$ then find 3A and – 2A.
2. Compute $A + B$ if defined for the following

   (i) $A=\begin{bmatrix}-1 & 4 & 7\\ 8 & 5 & 1\\ 2 & 8 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 1\\ 8 & 0 & 5\\ 3 & 2 & 4\end{bmatrix}$

   (ii) $A=\begin{bmatrix}-1 & 2 & 3\\ 4 & -5 & 6\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -3\\ 0 & 5 & 0\end{bmatrix}$
3. If $\begin{bmatrix}\mathbf{1} & \mathbf{-5}\\ \mathbf{6} & \mathbf{7}\end{bmatrix}+\boldsymbol{X}=\begin{bmatrix}\mathbf{2} & \mathbf{-7}\\ \mathbf{1} & \mathbf{8}\end{bmatrix}$, then find X.
4. If $A=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-3}\\ \mathbf{5} & \mathbf{0} & \mathbf{2}\\ \mathbf{1} & \mathbf{-1} & \mathbf{1}\end{bmatrix}$, $B=\begin{bmatrix}\mathbf{3} & \mathbf{-1} & \mathbf{2}\\ \mathbf{4} & \mathbf{2} & \mathbf{5}\\ \mathbf{2} & \mathbf{0} & \mathbf{3}\end{bmatrix}$. Find a matrix C such that $A + 2C = B$.
5. If $2X+3Y=\begin{bmatrix}\mathbf{2} & \mathbf{3}\\ \mathbf{4} & \mathbf{0}\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}\mathbf{-2} & \mathbf{2}\\ \mathbf{1} & \mathbf{-5}\end{bmatrix}$. Find X and Y.
6. Find X and Y if $2X+Y=\begin{bmatrix}\mathbf{4} & \mathbf{4} & \mathbf{7}\\ \mathbf{7} & \mathbf{3} & \mathbf{4}\end{bmatrix}$ **and** $\boldsymbol{X}-\mathbf{2}\boldsymbol{Y}=\begin{bmatrix}\mathbf{-3} & \mathbf{2} & \mathbf{1}\\ \mathbf{1} & \mathbf{-1} & \mathbf{2}\end{bmatrix}$.
7. If $A=\begin{bmatrix}2 & 1 & 3\\ 4 & 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}0 & 5 & 1\\ 2 & 3 & 1\end{bmatrix}$ then find C if $A + B + C$ is zero matrix.

Answer.

1. (i) $\begin{bmatrix}6 & 9\\ 12 & 18\end{bmatrix}$ (ii) $\begin{bmatrix}-4 & -6\\ -8 & -12\end{bmatrix}$

2. (i) $\begin{bmatrix}1 & 7 & 8\\ 16 & 5 & 6\\ 5 & 10 & 9\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 3 & 0\\ 4 & 0 & 6\end{bmatrix}$

3. $X=\begin{bmatrix}1 & -2\\ -5 & 1\end{bmatrix}$

4. $C=\begin{bmatrix}1 & -3/2 & 5/2\\ -1/2 & 1 & 3/2\\ 1/2 & 1/2 & 1\end{bmatrix}$.

5. $X=\begin{bmatrix}-2 & 0\\ -1 & -3\end{bmatrix}$, $Y=\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}$.

6. $X=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$ and $Y=\begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

7. $\begin{bmatrix}-2 & -6 & -4\\ -6 & -5 & -6\end{bmatrix}$

3.4. Multiplication of Matrices.

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$ and $b = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}_{n\times p}$ be two matrices of orders $m \times n$ and $n \times p$ respectively.

Then their product AB is a matrix C of order $m \times p$ and can be obtained as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{in}b_{nj} \qquad \text{for} \qquad 1 \le i \le m,\ 1 \le j \le p$$

where $C = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}_{m\times p}$. For an example, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

Then AB can be obtained as follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ \rightarrow & \rightarrow \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \downarrow & b_{12} \downarrow & b_{13} \\ b_{21} \downarrow & b_{22} \downarrow & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}.$$

Remark. If the number of columns of A are not equal to number of rows of B then the product AB is not defined. For the above example, the product BA is not possible as number of columns in B (3) is not equal to number of rows in A (2).

3.4.1. Properties of matrix multiplication.

1. Matrix multiplication is not commutative in general, that is, AB may or may not equal to BA.
2. Matrix multiplication is associative, that is, if A, B and C are matrices of order $m \times n$, $n \times p$ *and* $p \times q$ *respectively, then* $(AB)C = A(BC)$.
3. Matrix multiplication is distributive over addition, that is, if A, B and C are matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively, then $A(B+C) = AB+AC$.
4. If A and B are n-rowed matrices then

   i) $(A+B)^2 = A^2 + B^2 + AB + BA$

   ii) $(A-B)^2 = A^2 + B^2 - AB - BA$

   iii) $(A+B)(A-B) = A^2 - AB + BA - B^2$

3.4.2. Matrices and Polynomials.

If $f(x)$ is a polynomial of degree n

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n.$$

and A is a square matrix of order m, then $f(A)$ is defined as:

$$f(A) = a_0 A^n + a_1 A^{n-1} + \ldots\ldots + a_{n-1} A + a_n I_m$$

3.4.3. Example. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$**, then Compute** ***AB*****.**

Solution. By definition of product of two matrices

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 2+6+12 & 5+12+21 & 3+8+15 \\ 8+15+24 & 20+30+42 & 12+20+30 \end{bmatrix} = \begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}.$$

3.4.4. Example. If $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$**, then find** ***x*** **and** ***y*****.**

Solution. Given that $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. Using the definition of product of two matrices

$$\begin{bmatrix} 3x-4y \\ x+2y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

On comparing the corresponding elements, we get

$$3x - 4y = 3$$

$$x + 2y = 11$$

Solving these two, we get $x = 5$ and $y = 3$.

3.4.5. Example. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$**A =** $\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$ **then find** ***A*****.**

Solution. Since the product matrix is a 3×3 matrix and the first matrix in product is of order 3×2. Therefore, A must be a 2×3 matrix. So let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then, the given matrix equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

which implies $\begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

Comparing corresponding elements, we get

$$2a-d = -1,\ a = 1,\ -3a + 4d = 9,$$
$$2b-e = -8,\ b = -2,\ -3b + 4e = 22,$$
$$2c-f = 10,\ c = -5,\ -3c + 4f = 15.$$

Solving these equations, we get

$$a = 1, d = 3;\ b = -2, e = 4;\ c = -5, f = 0.$$

Hence, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

3.4.6. Example. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$ and hence evaluate A^4.

Solution. Here, $A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} . \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3.3+1.(-1) & 3.1+1.2 \\ -1.3+2.(-1) & -1.1+2.2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$.

Thus, $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $A^2 - 5A + 7I = O$

Now, $A^2 = 5A - 7I \quad \Rightarrow \quad A^4 = A^2.A^2 = (5A - 7I).(5A - 7I) = 25A^2 - 35A - 35A + 49I$

$$= 25(5A - 7I) - 70A + 49I = 125A - 175I - 70A + 49I$$

$$= 55A - 126I$$

$$= 55\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 126\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 165 & 55 \\ -55 & 110 \end{bmatrix} - \begin{bmatrix} 126 & 0 \\ 0 & 126 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}.$$

3.4.7. Example. A trust fund has Rs. 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of Rs. 1800.

Solution. Total fund is Rs. 30,000. Let investment made in first bond is Rs. x, then investment in the second bond Rs. $30{,}000 - x$. As per given data

Annual interest on first bond = **5% =** $\frac{5}{100}$ per rupee.

Annual interest on second bond = 7% = $\frac{7}{100}$ per rupee.

Let A be the investment matrix, then A = $[x \quad 30{,}000 - x]$, and B be the annual interest per rupee matrix, thus B = $\begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}$.

Therefore, total annual interest can be obtained by AB = $[x \quad 30{,}000 - x] \begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}$

$$= \left[\frac{5x}{100} + \frac{7(30000 - x)}{100} \right]$$

$$= \left[\frac{5x + 210000 - 7x}{100} \right] = \left[\frac{210000 - 2x}{100} \right]$$

Therefore, total annual interest is $\frac{210000-2x}{100}$.

Now for a total annual interest of Rs. 1800 , we must have

$$\frac{210000-2x}{100}=1800$$

which implies, $210000-2x=180000$

which implies, $2x=30000$

which implies, $x=15000$

Hence the investments in both bonds are Rs. 15000.

3.4.8. Exercise.

1. Find the order of $A\times B$ if the order of A and B are

   (i) 2×3 and 3×2 (ii) 5×4 and 4×3.

2. If $A=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}a & 0\\ b & -1\end{bmatrix}$ and $(A+B)^2=A^2+B^2$. Find a and b.

3. Give examples of matrices

   (i) A and B such that $AB=O$, but $A\neq 0$, $B\neq 0$

   (ii) A, B, C, such that $AB=AC$, but $B\neq C$; $A\neq 0$

4. If $A=[1\ \ 2\ \ 3]$, $B=\begin{bmatrix}2 & 1\\ 0 & 3\\ 5 & 4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2\\ 3 & 2\\ 1 & 0\end{bmatrix}$. Verify that $A(B+C)=AB+AC$.

5. Solve the matrix equation $[x\ \ 1]\begin{bmatrix}1 & 0\\ -2 & -3\end{bmatrix}\begin{bmatrix}x\\ 5\end{bmatrix}=0$.

6. If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$, then find k so that $A^2=8A+kI_2$.

7. A bookseller furnished a school with 10 dozen books for class X, 8 dozen books for class XI and 5 dozen books for class XII. If their prices are Rs. 83, Rs. 34.50 and Rs. 45 respectively per book, find the total amount of the bill furnished by the bookseller.

8. There are three families. Family A consists of 2 men, 3 women and 1 child. Family B has 2 men, 1 woman and 3 children. Family C has 4 men, 2 women and 6 children. Daily income of a man and woman are Rs. 20 and Rs. 15.50 respectively and children have no income. Using matrix multiplication, calculate the daily income of each family.

Answers.

1. (i) 2×2 (ii) 5×3. **5.** $x=5$ or $x=-3$. **6.** $k=-7$. **7.** Rs. 15972

8. Rs. 86.50, Rs. 55.50, Rs. 111

3.5. Transpose of a Matrix.

Let A be any $m \times n$ matrix, then any $n \times m$ matrix obtained from A by changing its rows into columns or columns into rows is called the transpose of A and is denoted by A^T or A'.

For example, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3}$ then $A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}_{3\times 2}$.

3.5.1. Properties of Transpose.

1. $(A+B)' = A' + B'$ (A and B being same order)
2. $(A')' = A$
3. $(k\,A)' = k\,A'$ (where k is any scalar)
4. $(AB)' = B'A'$ (A and B being conformable for product).

3.6. Symmetric and Skew Symmetric Matrices.

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ is called a **symmetric matrix** if $A' = A$ or $a_{ij} = a_{ji}$ for all i and j.

For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 8 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 8 \end{bmatrix}$**, then** $A' = A$ and hence A is a symmetric matrix.

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ is called a **skew- symmetric matrix** if $A' = -A$ or $a_{ij} = -a_{ji}$ for all i and j.

For example, let $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ then $A' = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix} = -A$. Hence A is skew-symmetric matrix.

3.6.1. Theorem. The diagonal elements of a skew symmetric matrix are all zero.

Proof. Let $A = \left[a_{ij}\right]$ be a skew symmetric matrix. Then by definition of skew symmetric matrix, $a_{ij} = -a_{ji}$ for all i, j. However, for diagonal elements $i = j$,

$$\Rightarrow \qquad a_{ii} = -a_{ii} \text{ for all values of } i$$

$$\Rightarrow \qquad 2a_{ii} = 0 \text{ for all } i\,.$$

$$\Rightarrow \qquad a_{ii} = 0 \text{ for all } i$$

Hence diagonal elements of a skew symmetric matrix are all zero.

3.6.2. Theorem. If A is any square matrix , then

1. $\frac{1}{2}(A + A')$ is symmetric matrix.
2. $\frac{1}{2}(A - A')$ is skew symmetric.
3. AA' and $A'A$ are symmetric matrix.

Proof.

1. Let $K = \frac{1}{2}(A + A')$, then as $(A+B)' = A' + B'$ and $(A')' = A$, therefore

$$K' = \frac{1}{2}(A + A')' = \frac{1}{2}\left[A' + (A')'\right] = \frac{1}{2}(A' + A) = \frac{1}{2}(A + A') = K$$

So, $K = \frac{1}{2}(A + A')$ is a symmetric matrix.

2. Let $K = \frac{1}{2}(A - A')$, then

$$K' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}\left[A' - (A')'\right] = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -K$$

Hence, $K = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

3. Let $K = AA'$, then as $(AB)' = B'A'$, so

$$K = (AA')' = (A')'A' = AA' = K.$$

Hence K is symmetric.

Similarly $A'A$ is symmetric.

3.6.3. Theorem. A matrix which is both symmetric and skew symmetric, must be a null matrix.

Proof. Let A be a matrix which is both symmetric and skew symmetric. Then

$$A' = A \quad \text{and} \quad A' = -A$$

Subtracting these two, we obtain

$$2A = \mathbf{O}$$

which implies that A is a zero matrix

3.6.4. Theorem. Let *A* and *B* be symmetric matrices of same order, then

1. $A+B$ is a symmetric matrix.

2. $AB + BA$ is a symmetric matrix.

3. $AB - BA$ is a skew-symmetric matrix.

Proof.

Since A and B are symmetric matrices. So $A' = A$ and $B' = B$. Then, we have

1. $$(A + B)' = A' + B' = A + B$$

Hence A + B is symmetric.

2. $$(AB + BA)' = (AB)' + (BA)' = B'A' + A'B' = BA + AB = AB + BA.$$

Hence $AB + BA$ is symmetric matrix.

3. $(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$

Hence $AB - BA$ is skew symmetric matrix.

3.6.5. Theorem. Every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrix.

Proof. Let A be a square matrix , then we can write

$$A = \frac{1}{2}(A+A) = \frac{1}{2}(A + A' + A - A') = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q, \text{ say}$$

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$. As proved earlier P is a symmetric matrix and Q is a skew symmetric matrix.

To prove the uniqueness. If possible, assume that

$$A = B + C \qquad (1)$$

where R is symmetric and S is skew symmetric.

Then, $$A' = (B + C)' = B' + C' = B - C \qquad (2)$$

Adding (1) and (2),

$$A + A' = 2B \qquad \Rightarrow \qquad B = \frac{1}{2}(A + A') = P.$$

Subtracting (2) from (1) ,

$$A + A' = 2C \qquad \Rightarrow \qquad C = \frac{1}{2}(A - A') = Q.$$

Hence A is uniquely expressed as a sum of symmetric and skew symmetric matrix.

3.6.6. Example. Find the transpose of matrix $A = \begin{bmatrix} 3 & 4 & 7 \\ 1 & 2 & 5 \\ -3 & 4 & 5 \end{bmatrix}$.

Solution. By definition, transpose of $A = A' = \begin{bmatrix} 3 & 1 & -3 \\ 4 & 2 & 4 \\ 7 & 5 & 5 \end{bmatrix}$.

3.6.7. Example. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$**, then verify that** $(AB)' = B'A'$.

Solution. Given that $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Therefore, $AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 7 \\ 6 & 1 & -1 \end{bmatrix}$. Thus,

$$(AB)' = \begin{bmatrix} 8 & 6 \\ 3 & 1 \\ 7 & -1 \end{bmatrix}$$

Also, $B'A' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}' \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 3 & 1 \\ 7 & -1 \end{bmatrix}$.

So, we observed that

$$(AB)' = B'A'$$

3.6.8. Example. Express matrix A= $A = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix}$ as a sum of symmetric and skew symmetric matrices.

Solution. We know that $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew symmetric.

Now , $A + A' = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 10 & 18 & 3 \\ 7 & 4 & 1 \\ 9 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 20 & 25 & 12 \\ 25 & 8 & -9 \\ 12 & -9 & 14 \end{bmatrix}$

and $A - A' = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 10 & 18 & 3 \\ 7 & 4 & 1 \\ 9 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -11 & 6 \\ 11 & 0 & -11 \\ -6 & 11 & 0 \end{bmatrix}$.

Now $\frac{1}{2}(A + A') = \begin{bmatrix} 10 & \frac{25}{2} & 6 \\ \frac{25}{2} & 4 & -\frac{9}{2} \\ 6 & -\frac{9}{2} & 7 \end{bmatrix}$ which is symmetric.

and $\frac{1}{2}(A - A') = \begin{bmatrix} 0 & -\frac{11}{2} & 3 \\ \frac{11}{2} & 0 & -\frac{11}{2} \\ -3 & \frac{11}{2} & 0 \end{bmatrix}$ which is a skew symmetric matrix.

Thus $A = A = \begin{bmatrix} 10 & \frac{25}{2} & 6 \\ \frac{25}{2} & 4 & -\frac{9}{2} \\ 6 & -\frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{11}{2} & 3 \\ \frac{11}{2} & 0 & -\frac{11}{2} \\ -3 & \frac{11}{2} & 0 \end{bmatrix}$.

3.6.9. Exercise.

1. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -3 \\ 2 & 5 & 3 \end{bmatrix}$ verify that $(A+B)' = A' + B'$.
2. Find the transpose of following matrices

   (i) $A = \begin{bmatrix} 5 & 2 & 0 \\ 1 & 4 & 7 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 11 & 9 \\ 2 & 7 & 4 \\ 5 & 6 & 7 \end{bmatrix}$
3. Find values of x, y, z for the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ if $A'A = I$.
4. Express the following matrices as the sum of symmetric and skew symmetric matrix.

   (i) $\begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 3 \\ 0 & 6 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 7 \\ 6 & 5 & -8 \\ 1 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$

3.7. Check Your Progress.

4. Give an example of a matrix which is row matrix as well as column matrix.
5. Find a matrix X such that $2A + B + X = 0$ where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.
6. If $A = \begin{bmatrix} 2 & -8 \\ 5 & -3 \end{bmatrix}$, then show that $A + A'$ is symmetric and $A - A'$ is skew symmetric.

Answers.

1. Any square matrix of order 1 is a matrix which is both row matrix as well as column matrix.
2. $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$.

3.8. Summary. In this chapter, we discussed about Matrices, its various types, when we can add or subtract or multiply two matrices. In all cases the most important aspect is the order of the given matrices. Further, it was observed that ant square matrix can be expressed as sum of a symmetric and a skew-symmetric matrix.

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